

# Mathematical Justification of the Aharonov-Bohm Hamiltonian

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Received: 9 June 2008 / Accepted: 2 October 2008 / Published online: 11 November 2008  
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**Abstract** It is presented, in the framework of nonrelativistic quantum mechanics, a justification of the usual Aharonov-Bohm hamiltonian (with solenoid of radius greater than zero). This is obtained by way of increasing sequences of finitely long solenoids together with a natural impermeability procedure; further, both limits commute. Such rigorous limits are in the strong resolvent sense and in both  $\mathbb{R}^2$  and  $\mathbb{R}^3$  spaces.

**Keywords** Aharonov-Bohm hamiltonian · Finite solenoid · Strong resolvent convergence

Given a cylindrical current-carrying solenoid  $S$  of infinite length and radius  $a > 0$ , centered at the origin and axis in the  $z$ -direction, there is a constant magnetic field  $\mathbf{B} = (0, 0, B)$  confined in  $S^\circ$ , the interior of  $S$ , and vanishing in its exterior region  $S'$ . The solenoid is considered impermeable (impenetrable), in the sense that the motion of a spinless particle (of mass  $m = 1/2$  and electric charge  $q$ ) outside the solenoid has no contact with its interior, particularly with the magnetic field  $\mathbf{B}$ . If  $\mathbf{A}$  is the vector potential generating this magnetic field, that is,  $\mathbf{B} = \nabla \times \mathbf{A}$ , the usual hamiltonian operator describing the quantum motion of this charged particle is given by (with  $\hbar = 1$ )

$$H_{AB} = \left( \mathbf{p} - \frac{q}{c} \mathbf{A} \right)^2, \quad \mathbf{p} = -i\nabla,$$

with Dirichlet boundary conditions, i.e., the functions  $\psi$  in the domain of  $H_{AB}$  are supposed to vanish  $\psi = 0$  at the solenoid boundary (the precise domain of  $H_{AB}$  is described just before Proposition 1). Observable effects, as wavefunction phase differences, are predicted and confirmed in many experiments, even though the particle is confined to  $S'$  (see the original paper [2–4] and [15, 24, 26, 27, 30, 31] for detailed descriptions and a long list of additional references). Since the vector potential is not assumed to (identically) vanish in the

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exterior region  $S'$ , the interpretation in [2–4], and followed by a huge amount of papers, is that  $\mathbf{A}$  plays a prominent role in quantum mechanics, so that these measurable effects would be regarded to be caused exclusively to  $\mathbf{A}$ , and not just to the magnetic field  $\mathbf{B}$ . Since then, this has been called *the Aharonov-Bohm effect* (in spite of such question had been considered previously [14, 16]) and it is directly related to the acceptance of  $H_{AB}$  above as the quantum model of such situation (particularly, the presence of the vector potential in the hamiltonian).

Note that usually the papers devoted to the Aharonov-Bohm effect in different contexts simply accept the above hamiltonian operator  $H_{AB}$  prescription (suitably adapted; e.g., two solenoids) and interpretations. The goal of this communication is to comment on the difficulties in the quantization process in this setting and, mainly, to give grounds for  $H_{AB}$  from a combination of physical modeling and precise mathematical arguments. There are other (although related) versions of the Aharonov-Bohm effect (e.g., with electric potentials), but the above is the most considered and traditional one; furthermore, many works consider the idealized case of a solenoid of radius zero (for instance [1–4, 12, 30], to mention a few), but here we concentrate on the more realistic case of radius  $a > 0$ .

There are controversies over the interpretation of  $\mathbf{A}$  as a real physical variable, that is, mistrusts of the existence of the Aharonov-Bohm effect as stated above. For instance, that the phase difference could be eliminated by using gauge transformations [8, 9]; explanation via the hydrodynamical viewpoint in quantum mechanics [10], whose equations admit a solution where the vector potential appears explicitly, and such solution corresponds to a hamiltonian with the vector potential included; some authors argue that the experimental results could be explained by a border effect and the magnetic field (also due to poor solenoid impermeability) in a region accessible to the electric particles [18, 29]—see comments and critiques in [17, 21, 23].

The acceptance of  $H_{AB}$ , with the explicit appearance of nonzero  $\mathbf{A}$  outside the impermeable solenoid, even though  $\mathbf{B} = \nabla \times \mathbf{A} = 0$  there, is primarily based on an application of Stokes theorem: if  $C$  is a (closed) loop in  $S'$  around the solenoid, enclosing an area  $\mathcal{A}$ , then it is assumed that the total magnetic flux crossing  $\mathcal{A}$  is

$$\Phi = \int_{\mathcal{A}} \mathbf{B} \cdot d\mathbf{a} = \oint_C \mathbf{A} \cdot d\mathbf{l},$$

and it is argued that a phase difference should be expected between paths from the left and right pieces of  $C$  (if  $\alpha = q\Phi/(2\pi c)$  is not an integer number). However, this argument presents mathematical and physical difficulties that should be carefully justified. From the mathematical point of view it involves Stokes theorem in a multiply connected domain and its application is not guaranteed (although it is if  $\mathcal{A}$  does not intersect the interior  $S^\circ$ ); from the physical point of view the assumed (electromagnetic) impermeability of the solenoid should, in principle, inhibit also any nonzero vector potential in  $S'$  from sources in  $S^\circ$ . In summary, the acceptance of  $H_{AB}$  involves an explicitly choice that needs an explanation (without mention the Dirichlet boundary conditions; see ahead). Actually, this is a reflection of the fact that quantization in multiply connected domains is not a well-posed question (and here with some structure  $\mathbf{B} \neq 0$  inside the hole!). Clearly, geometrical and topological aspects have also been invoked to study the Aharonov-Bohm effect (see, e.g., [19, 25]). The effect of an electric field induced by a slowly switching on flux inside the solenoid was studied by Weisskopf [32] in 1961.

In what follows we present a justification of the hamiltonian  $H_{AB}$ . We propose to consider first a solenoid  $S_L$  of finite length  $2L > 0$  and also permeable. Recall that a current-carrying finite solenoid generates a nonzero magnetic field in its exterior and, since it is also considered permeable, Stokes theorem may be applied; therefore the corresponding hamiltonian

operator is well defined and with no boundary condition at the solenoid border. We model the impermeability by a sequence of positive potentials  $V_n$  which vanish in the solenoid exterior  $S_L^c$  region and goes to infinity in its interior  $S_L^\circ$  as  $n \rightarrow \infty$  [24] (to the best of authors knowledge the first one to propose the solenoid impermeability via increasing potentials  $V_n$  was Kretzschmar [22]). Then we discuss the limits of solenoid of infinite length, i.e.,  $L \rightarrow \infty$ , and impermeability  $n \rightarrow \infty$  (so getting a multiply connected region) by showing they exist (in the resolvent sense [28]) and, finally, that both limits commute, that is, it does not matter which limit is taken first, and the resulting hamiltonian is always  $H_{AB}$ . Such limits are in the strong resolvent sense in  $\mathbb{R}^2$  and  $\mathbb{R}^3$  and we discuss both simultaneously, since the arguments are almost the same.

Few papers have explicitly considered a finite solenoid [5, 29] in this context; also some mathematically nonrigorous limiting process are discussed in [7] in order to justify the hamiltonian. However, the arguments may not be considered in the typical criteria of rigor of mathematical physics we demand here, and this is our main contribution. One difficulty is that the deficiency indices of the Aharonov-Bohm hamiltonian  $H_{AB}$  with domain  $C_0^\infty(S')$  are both infinite, which leads to a plethora of self-adjoint extensions; all self-adjoint extensions of this operator will appear elsewhere [13].

It is worth mentioning the experiments conducted by Tonomura and collaborators [27] with toroidal magnets, which have the advantage of no magnetic flux with leaks; recently a rigorous approach to the scattering in this case (and more general ones) has been done in [6].

Now we go into details of the idea sketched above for the justification of  $H_{AB}$ . Let  $x = (x_1, x_2, x_3)$  denote the cartesian coordinates in  $\mathbb{R}^3$ ; the interior of the finite solenoid  $S_L$ , symmetrically disposed with respect to the plane  $x_1, x_2$ , is

$$S_L^\circ = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 < a^2, |x_3| < L\},$$

and denote by  $\chi_L$  its characteristic function, that is,  $\chi_L(x) = 1$  if  $x \in S_L^\circ$  and  $\chi_L(x) = 0$  otherwise. The sequence of potential barriers will be  $V_n(x) = n\chi_L(x)$ . If  $\mathbf{A}_L$  denotes the vector potential generated by this finite permeable solenoid, then the corresponding hamiltonian is ( $0 < L \leq \infty$ ; note that we write  $\mathbf{A} = \mathbf{A}_{L=\infty}$  and  $S = S_{L=\infty}$ )

$$H_{L,n} = \left(\mathbf{p} - \frac{q}{c}\mathbf{A}_L\right)^2 + V_n, \quad \text{dom } H_{L,n} = \mathcal{H}^2(\mathbb{R}^d), \quad d = 2, 3,$$

where  $\mathcal{H}^2$  denotes the usual Sobolev space domain of the free hamiltonian (that is, the negative laplacian)  $-\Delta$ . In case of  $\mathbb{R}^2$  we just restrict the vector potential and  $V_n$  to the plane and  $S \cap \mathbb{R}^2$  is a disk centered at the origin.

In  $\mathbb{R}^3$  there is the possibility of the particle running into the finite solenoid at a point with  $x_3 = \pm L$  (with total area  $\alpha_t = 2\pi a^2$ ), which is physically different from entering through the lateral border of the solenoid (with area  $\alpha_l = 2\pi a \times (2L)$ ), but the potential barrier  $V_n$  equally hinders the entrance of the particle from any direction. This effect becomes less and less important as  $L$  increases, since the area ratio  $\alpha_t/\alpha_l \rightarrow 0$  as  $L \rightarrow \infty$  (note also that, in fact,  $\alpha_t$  does not depend on  $L$ ) and for large  $L$  the solenoid top and bottom will generally be far away from the electron motion; furthermore, this effect is not present in two-dimensions. Hence, it will not be modeled here.

In both dimensions  $d = 2, 3$ , the finiteness of the solenoid and permeability make the modeling more feasible, and for each finite-valued pair  $n, L$  the hamiltonian  $H_{L,n}$  is a well-posed operator and self-adjoint. Note that  $\mathbf{A}_L$  is a bounded and continuous vector function; for instance, in  $\mathbb{R}^2$ , by using cylindrical coordinates  $(\rho, \phi, z)$ ,  $z = 0$ , and the calculation in

[20] of the vector potential of a circular current loop, we find that (in a particular gauge) the  $\rho, z$  components of  $\mathbf{A}_L$  vanish, whereas the  $\phi$  component depends only on  $\rho$  and is given by

$$A_{L,\phi}(\rho) = \frac{\Phi}{4\pi^2 a} \int_{-L}^L dz' \int_0^{2\pi} d\phi' \frac{\cos \phi'}{(\rho^2 + a^2 + z'^2 - 2a\rho \cos \phi')^{1/2}}.$$

Now, for  $\rho \neq a$  (the solenoid border  $\rho = a$  is a set of zero Lebesgue measure), we have the expected pointwise convergence of  $\mathbf{A}_L$  to  $\mathbf{A}$  as  $L \rightarrow \infty$  in  $\mathbb{R}^2$ , whose  $\phi$  component of  $\mathbf{A}$  is well known and given by  $A_\phi(\rho) = \Phi/(2\pi\rho)$  if  $a \leq \rho$ , and  $A_\phi(\rho) = \Phi\rho/(2\pi a^2)$  if  $0 \leq \rho \leq a$ . Similarly for the pointwise convergence as  $L \rightarrow \infty$  of vector potentials in  $\mathbb{R}^3$ . See the [Appendix](#) for details.

In the particular case of an infinite length solenoid  $L = \infty$  in  $\mathbb{R}^3$ , the impermeable limit  $n \rightarrow \infty$  was considered in [24]; by using Kato-Robinson theorem [11] it was shown that  $H_{\infty,n}$  converges to  $H_{AB}$  with domain

$$\text{dom } H_{AB} = \mathcal{H}^2(S') \cap \mathcal{H}_0^1(S')$$

in the strong resolvent sense as  $n \rightarrow \infty$ , and since elements of  $\mathcal{H}_0^1(S')$  vanish at the solenoid border (in the sense of Sobolev traces), Dirichlet boundary conditions have showed up in this situation. Since the same procedure of [24] for impermeability applies to the case of finite solenoids  $S_L$  (with  $L$  fixed), we obtain (in dimensions  $d = 2, 3$ )

**Proposition 1** *As  $n \rightarrow \infty$  the operator sequence  $H_{L,n}$  converges in the strong resolvent sense to the operator*

$$H_{L,\infty} := \left( \mathbf{p} - \frac{q}{c} \mathbf{A}_L \right)^2, \quad \text{dom } H_{L,\infty} = \mathcal{H}^2(S'_L) \cap \mathcal{H}_0^1(S'_L).$$

Fix now  $n$ . If  $\psi \in C_0^\infty(\mathbb{R}^d)$  and  $\text{supp } \psi$  denotes its support, then

$$\|H_{L,n}\psi - H_{\infty,n}\psi\|^2 = \int_{\text{supp } \psi} |2i(\mathbf{A}_L - \mathbf{A}) \cdot \nabla \psi + (\mathbf{A}_L^2 - \mathbf{A}^2)\psi|^2 dx,$$

and since as  $L \rightarrow \infty$  we have the pointwise limit  $\mathbf{A}_L \rightarrow \mathbf{A}$ , it follows that  $H_{L,n}\psi \rightarrow H_{\infty,n}\psi$  by Lebesgue dominated convergence. Since the set  $C_0^\infty(\mathbb{R}^d)$  is a core of both  $H_{\infty,n}$  and  $H_{L,n}$ , for all  $L > 0$ , an application of Theorem VIII.25 of [28] implies

**Proposition 2** *For each fixed  $n$ , the operator sequence  $H_{L,n}$  converges to  $H_{\infty,n}$  in the strong resolvent sense as  $L \rightarrow \infty$ .*

Let  $R_i(T) = (T - i)^{-1}$  denote the resolvent of a self-adjoint operator  $T$  at the complex number  $i$ . For  $\psi \in L^2(S')$  we have

$$\begin{aligned} & \|R_i(H_{L,\infty})\psi - R_i(H_{AB})\psi\| \\ & \leq \|R_i(H_{L,\infty})\psi - R_i(H_{L,n})\psi\| + \|R_i(H_{L,n})\psi - R_i(H_{\infty,n})\psi\| \\ & \quad + \|R_i(H_{\infty,n})\psi - R_i(H_{AB})\psi\|, \end{aligned}$$

and, given  $\epsilon > 0$ , by Proposition 2, if  $L$  is large enough we have

$$\|R_i(H_{L,n})\psi - R_i(H_{\infty,n})\psi\| < \epsilon/3,$$

and after fixing such  $L$  we subsequently take  $n$  large enough so that, by Proposition 1 and the resolvent convergence  $H_{\infty,n} \rightarrow H_{AB}$  [24],

$$\|R_i(H_{L,\infty})\psi - R_i(H_{L,n})\psi\| < \epsilon/3, \quad \|R_i(H_{\infty,n})\psi - R_i(H_{AB})\psi\| < \epsilon/3,$$

respectively, so that

$$\|R_i(H_{L,\infty})\psi - R_i(H_{AB})\psi\| < \epsilon$$

for  $L$  large enough. We have proved:

**Proposition 3** *The operator  $H_{L,\infty}$  converges to  $H_{AB}$  in the strong resolvent sense as  $L \rightarrow \infty$ .*

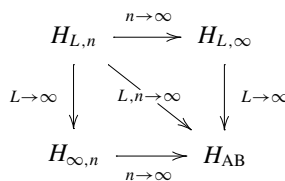
Let  $P_0$  denote the projection operator  $L^2(\mathbb{R}^d) \rightarrow L^2(S')$ . If  $\psi \in L^2(\mathbb{R}^d)$ , then

$$\begin{aligned} &\|R_i(H_{AB})P_0\psi - R_i(H_{L,n})P_0\psi\| \\ &\leq \|R_i(H_{AB})P_0\psi - R_i(H_{L,\infty})P_0\psi\| + \|R_i(H_{L,\infty})P_0\psi - R_i(H_{L,n})P_0\psi\|. \end{aligned}$$

By the above propositions both terms on the rhs vanish as  $L, n \rightarrow \infty$ , and so we conclude

**Theorem 1**  *$H_{L,n} \rightarrow H_{AB}$  in the strong resolvent sense as  $L, n \rightarrow \infty$ , independently of the way both limits are taken.*

See [11, 24] for a discussion of resolvent convergence when the domain of the limit operator is not dense in the original space (as  $L^2(S')$  is not dense in  $L^2(\mathbb{R}^d)$ ). Theorem 1 says that the same operator  $H_{AB}$  is obtained independently of the way the limits of infinitely long solenoid and impermeability are processed. For instance, both operations could be done simultaneously by taken, say,  $n = L$  and then  $L \rightarrow \infty$ , etc. In particular, the limits  $L \rightarrow \infty$  and  $n \rightarrow \infty$  do commute. This is summarized in the diagram ahead.



Therefore, we are justified in using  $H_{AB}$  while modeling an infinitely long and impermeable solenoid, even though we are in a situation of multiply connectedness and with a magnetic field restricted to the (impenetrable) region.

*Remark 1* a) For each fixed  $n$  it is possible to check that  $H_{L,n}$  converges to  $H_{\infty,n}$  in the strong sense in  $\mathcal{H}^2(\mathbb{R}^d)$  as  $L \rightarrow \infty$ , for  $d = 2, 3$ .

b) By using different techniques, for  $d = 2, 3$  it is possible to show that for each  $L < \infty$  fixed,  $H_{L,n}$  converges to  $H_{L,\infty}$  in the uniform resolvent sense as  $n \rightarrow \infty$ . This uniform convergence also holds for  $L = \infty$  in  $\mathbb{R}^2$ ; however, such uniform convergence should not be expected to occur in  $\mathbb{R}^3$  when  $L = \infty$ , because the solenoid border is not compact in this case.

The limit procedures discussed here constitute a step further and complementary to [24], which has considered only infinitely long solenoids.

It is intriguing that the (usually just formal) convergence of the limiting processes to  $H_{AB}$  has led different authors to extremely opposite conclusions: whereas Magni and Valz-Gris [24, pp. 185–186] concluded that “The way of coming to that hamiltonian, however, makes it clear that there is no cogent reason to attribute vector potentials any physical activity. . . .” Berry [7] argues that such limits justify the exclusive quantum role of potentials. At least with respect to this work, we decided to keep back from this controversy and restrict ourselves to the above diagram.

**Acknowledgements** The authors acknowledge partial support from CNPq (Brazil).

### Appendix

In this appendix we find the expression of the vector potential  $\mathbf{A}_L$  generated by a finite solenoid of length  $2L$  in  $\mathbb{R}^3$ , in a suitable gauge. Then we show its pointwise convergence to  $\mathbf{A}$  as  $L \rightarrow \infty$ . Everything works in the plane  $\mathbb{R}^2$ . This fact was used in the proofs above.

#### Vector Potential of a Finite Solenoid

The starting point is the vector potential due to a circular current loop performed in [20, Sect. 5.5]. Then an integration over a density of loops gives the desired vector potential. Consider a circular loop of radius  $a > 0$  centered at (cartesian coordinates)  $(0, 0, z')$ ,  $z' \geq 0$ , and parallel to the plane  $xy$ . Let  $\mathbf{x}'$  be a point of the loop and  $\mathbf{x}$  a general point in  $\mathbb{R}^3$ , whose spherical coordinates are  $\mathbf{x}' = (r', \theta', \phi')$  and  $\mathbf{x} = (r, \theta, \phi)$ , respectively.

The only nonzero component of the current density  $\mathbf{J}$  is in the  $\phi$  direction and, by following [20], it is given by

$$J_\phi = I \delta\left(\cos\theta' - \frac{z'}{\sqrt{a^2 + z'^2}}\right) \frac{\delta(r' - \sqrt{a^2 + z'^2})}{\sqrt{a^2 + z'^2}},$$

with  $I$  denoting the loop electric current. Due to the symmetry of the problem, it is possible to assume that the resulting vector potential has only component the  $\phi$  direction, which actually does not depend on  $\phi$ ; then select  $\phi = 0$  in the computation that follows. One has

$$A_\phi^{z'}(r, \theta) = \frac{I}{c\sqrt{a^2 + z'^2}} \int r'^2 dr' d\Omega' \frac{\cos\phi' \delta(\cos\theta' - \frac{z'}{\sqrt{a^2 + z'^2}}) \delta(r' - \sqrt{a^2 + z'^2})}{|\mathbf{x} - \mathbf{x}'|},$$

with  $|\mathbf{x} - \mathbf{x}'| = [r^2 + r'^2 - 2rr'(\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos\phi)']^{1/2}$  and  $d\Omega' = \sin\theta' d\theta' d\phi'$ .

On integrating with respect to  $r' = \sqrt{a^2 + z'^2}$ , and then with respect to  $\theta'$ , with  $\cos\theta' = \frac{z'}{r'}$  and  $\sin\theta' = \frac{a}{r'}$ , one finds

$$A_\phi^{z'}(r, \theta) = \frac{I}{c\sqrt{a^2 + z'^2}} \int_0^{2\pi} (a^2 + z'^2) \frac{a}{\sqrt{a^2 + z'^2}} d\phi' \frac{\cos\phi'}{|\mathbf{x} - \mathbf{x}'|},$$

that is,

$$A_{\phi}^{z'}(r, \theta) = \frac{Ia}{c} \int_0^{2\pi} d\phi' \frac{\cos \phi'}{[r^2 + a^2 + z'^2 - 2r\sqrt{a^2 + z'^2}(\cos \theta \frac{z'}{\sqrt{a^2+z'^2}} + \sin \theta \frac{a}{\sqrt{a^2+z'^2}} \cos \phi')]^{1/2}}.$$

Similarly for  $z' \leq 0$ .

Thus, the vector potential of the finite solenoid of length  $2L$  at a point  $\mathbf{x} = (r, \theta, \phi)$  in spherical coordinates is  $\mathbf{A}_L = (0, 0, A_{L,\phi})$ , where

$$A_{L,\phi}(r, \theta) = n \int_{-L}^L dz' A_{\phi}^{z'}(r, \theta) = \frac{\Phi}{4\pi^2 a} \int_{-L}^L dz' \int_0^{2\pi} d\phi' \frac{\cos \phi'}{f(r, \theta, z', \phi')},$$

and  $n$  is the number of loops by length unit in the solenoid,  $\Phi$  the magnetic flux (so that  $\frac{\Phi}{4\pi^2 a} = \frac{nIa}{c}$ ) and, finally,

$$f(r, \theta, z', \phi') := (r^2 + a^2 + z'^2 - 2rz' \cos \theta - 2ra \sin \theta \cos \phi')^{1/2}.$$

Here we use the notation  $A_{\phi}(r, \theta) = A_{\infty,\phi}(r, \theta)$  for the  $\phi$  component of the vector potential in case  $L = \infty$ .

Note that for  $\theta = \pi/2$  we have  $z = 0$  and we obtain the vector potential in a point  $\mathbf{x} = (r, \pi/2, \phi)$  of the  $xy$  plane

$$A_{L,\phi}\left(r, \frac{\pi}{2}\right) = \frac{\Phi}{4\pi^2 a} \int_{-L}^L dz' \int_0^{2\pi} d\phi' \frac{\cos \phi'}{(r^2 + a^2 + z'^2 - 2ra \cos \phi')^{1/2}},$$

which in polar coordinates was denoted by  $A_{L,\phi}(\rho)$  above. It can also be expressed in terms of complete elliptic integrals  $K(k)$  e  $E(k)$  [5], that is,

$$A_{L,\phi}(\rho) = \frac{\Phi}{\pi^2 a} \int_{-L}^L dz' \frac{(2 - k^2)K(k) - 2E(k)}{k^2[(a + \rho)^2 + z'^2]^{1/2}},$$

and  $k$  is given by  $k^2 = 4a\rho/[(a + \rho)^2 + z'^2]$ .

*Convergence as  $L \rightarrow \infty$*

Fix  $r, \theta$ . In three situations the term

$$\left| \frac{2ra \sin \theta \cos \phi'}{r^2 + a^2 + z'^2 - 2rz' \cos \theta} \right|$$

is uniformly small: either 1)  $r \ll a$ , or 2)  $r \gg a$  and  $r \gg 1$  or 3) for large  $z' \gg 1$  and  $z' \gg a$ . In any of such situations one has

$$\frac{\cos \phi'}{f(r, \theta, z', \phi')} = \frac{\cos \phi'}{g(r, \theta, z')^{1/2}} + \frac{ra \sin \theta \cos^2 \phi'}{g(r, \theta, z')^{3/2}} + \frac{3}{2} \frac{(ra \sin \theta)^2 \cos^3 \phi'}{g(r, \theta, z')^{5/2}} + O(r^{-4}, z'^{-7}),$$

with  $g(r, \theta, z') := r^2 + a^2 + z'^2 - 2rz' \cos \theta$ . Note that the integrals of the first and third terms on the rhs above vanish. Then, the error in the approximation of  $A_\phi(r, \theta)$  by  $A_{L,\phi}(r, \theta)$  can be estimated by (for  $L$  large)

$$\begin{aligned} |A_\phi(r, \theta) - A_{L,\phi}(r, \theta)| &= \left| \left( \int_L^\infty + \int_{-\infty}^{-L} \right) dz' A_{z'}^\phi(r, \theta) \right| \\ &= \left| \left( \int_L^\infty + \int_{-\infty}^{-L} \right) dz' \int_0^{2\pi} d\phi' \frac{\cos \phi'}{f(r, \theta, z', \phi')} \right| \\ &\leq \text{cte} \left| \int_L^\infty dz' \int_0^{2\pi} d\phi' \frac{ra \sin \theta \cos^2 \phi'}{g(r, \theta, z')^{3/2}} \right| \\ &\leq \text{cte} \int_L^\infty dz' \frac{1}{(z'^2 - 2rz')^{3/2}} \leq \text{cte} \int_L^\infty dz' \frac{1}{z'^3} = \frac{\text{cte}}{L^2}, \end{aligned}$$

which vanishes as  $L \rightarrow \infty$ . Note that we have got an upper bound to the rate of convergence as  $L^{-2}$  (this rate was also found numerically).

Now we check that the above expressions for  $A_{L,\phi}(r, \theta)$  actually result in the right gauge in the limit  $L \rightarrow \infty$ , that is, in cylindrical coordinates  $\rho = r \sin \theta$ ,

$$A_\phi(r, \theta) = \begin{cases} \Phi/(2\pi\rho) & \rho \geq a > 0 \\ \Phi\rho/(2\pi a^2) & 0 \leq \rho \leq a. \end{cases}$$

For this it is enough to calculate the vector potential for some range of  $r, \theta$ , say  $r \gg a$  and  $r \ll a$ .

Let's consider the case of a point  $\mathbf{x}$  far from the solenoid, that is,  $r \sin \theta \gg a$ . Substitute the above expression for  $\cos \phi'/f(r, \theta, z', \phi')$  in  $A_{L,\phi}(r, \theta)$  so that, after performing the resulting integrals,

$$A_{L,\phi}(r, \theta) \approx \frac{\Phi}{2\pi} \frac{r \sin \theta}{r^2 + a^2 - r^2 \cos^2 \theta} \frac{\alpha(L - r \cos \theta) + \beta(L + r \cos \theta)}{2\beta\alpha},$$

with  $\alpha = \sqrt{r^2 + a^2 + L^2 + 2r \cos \theta L}$  and  $\beta = \sqrt{r^2 + a^2 + L^2 - 2r \cos \theta L}$ . Hence, for large  $L$

$$A_{L,\phi}(r, \theta) \approx \frac{\Phi}{2\pi} \frac{r \sin \theta}{r^2 + a^2 - r^2 \cos^2 \theta}.$$

Taking into account that  $r \sin \theta \gg a$  again, we see that  $A_{L,\phi}$  approaches  $A_\phi$  above as  $L \rightarrow \infty$ , and the right gauge is obtained. Similar arguments hold for  $r \sin \theta \ll a$ . Observe that for  $\theta = \pi/2$  the above steps infer the convergence in the  $xy$  plane, that is, in  $\mathbb{R}^2$ .

We underline that for points  $\mathbf{x} \notin \mathcal{S}$  the integrand in the expression for  $A_{L,\phi}$  is a continuous function and, for fixed  $r, \theta$ , with  $r \sin \theta \neq a$ , there is  $d > 0$  so that the absolute value of the denominator in the integrand is uniformly  $\geq d$ . In fact, one can take

$$d := \min_{\mathbf{x}' \in \mathcal{S}} |\mathbf{x} - \mathbf{x}'| = |r \sin \theta - a| > 0.$$

In summary, off the solenoid border, the above expressions for the vector potentials result in finite values for both  $L < \infty$  and  $L = \infty$ .

For points  $\mathbf{x}$  on the solenoid border, that is,  $|\mathbf{x} - \mathbf{x}'| = 0$ , for some  $\mathbf{x}' \in \mathcal{S}$ , the denominator of the integrand in the expression for  $A_{L,\phi}$  vanishes, which causes a divergence in



the integrals; however, such expression for  $A_{L,\phi}$  is not supposed to hold on this border, and the values of  $\mathbf{A}$  are recovered by continuity (by using lateral limits from inside and outside of the solenoid). In any event, the solenoid border is a set of zero Lebesgue measure in  $\mathbb{R}^3$  and  $\mathbb{R}^2$ .

Finally, note that it is not necessary to consider a finite solenoid with  $-L < z' < L$ , since all arguments are easily adapted to  $-L_1 < z' < L_2$ , with  $L_1 \rightarrow \infty$ ,  $L_2 \rightarrow \infty$ .

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